Direct Digital Control of Linear Processes Subject to Random Disturbances

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The implementation of digital computers in direct digital control (DDC) of chemical processes is becoming increasingly important. For the control of systems subject to stochastic disturbances and control effort constraints, various approaches have been used. A convenient and often-used method of designing optimal controllers for linear time-invariant systems with random inputs (disturbances) is the so-called "Wiener-Hopf technique" (Wiener, 1949) which employs variational calculus in the frequency domain. Previous work in the application of Wiener-Hopf techniques to the design of DDC algorithms (Tou and Kumar, 1961; Chang, 1961), although substantial, lacks generality in areas which are important in process control. First of all, there are nonminimum phase (NMP) processes characterized by right half s-plane zeroes in the transfer function, such as steam boilers (Iinoya and Altpeter, 1962), distillation columns (Wood, 1967), absorbers (Lees, 1970), and chemical reactors (Lim and Bankoff, 1970). Also, many process systems exhibit dead time. Thirdly, unstable dynamics may occur, an example of which is the unstable stirred-tank chemical reactor (Aris, 1965). Tou and Kumar (1961) do not allow for differences in the plant and disturbance transfer functions; thus, their methods may not apply to systems with a NMP zero or dead time in the plant which is not present in the disturbance. This is a particularly serious restriction in view of the common occurrence of dead time in process models. Tou and Kumar (1961) also do not allow for any instability in the system, while Chang's (1961) claim for unstable process is valid only for servo problems.

The purpose of this note is to obtain results for digital controllers with the same degree of generality as those of Lim and Bankoff (1970a) which allow for instability, NMP zeroes or dead time, and to obtain valuable information concerning the implementation of such controllers on a particular chemical process.

CONTROL PROBLEM FORMULATION

The control system considered in this work is shown in Figure 1. The process itself is represented by

$$C(s) = G_u(s)\ U(s) + G_s(s)\ M(s) \qquad (1)$$

where

$$G_u(s) = e^{-\tau_u s} \sum_{i=0}^{j} \beta_i s^i \Big/ \sum_{i=0}^{n} \alpha_i s^i;$$

$$G_s(s) = e^{-\tau_s s} \sum_{i=0}^{k} \gamma_i s^i \Big/ \sum_{i=0}^{n} \alpha_i s^i$$

The output and the set point $c(t)$ and $r(t)$ are measured and $m_c(t)$ evaluated at discrete intervals with sampling period T. The quantities $D_r(z)$ and $D_u(z)$ are the controller pulse-transfer functions operating on $r(t)$ and $e(t)$ respectively. The zero-order hold is denoted by $G_h(s)$. The actual digital computer that would be used to determine $m_c(t)$ in a physical situation is indicated by the closed dashed line in Figure 1. The quantities $G_c(s)$ and $G_f(s)$ are auxiliary feedback and feedforward controllers, respectively, and may be required in certain situations. The manipulated variable $m(t)$ is made up of signals from all the different controllers. The constraint is always applied to $m(t)$. In the great majority of cases, however, $G_c(s)$ and $G_f(s)$ are absent so that $m(t) = m_c(t)$.

Since $u(t)$ is a random or stochastic signal, its value at a given instant in time is not known. Thus, controller design must be based on the statistical properties of $u(t)$ rather than on $u(t)$ iself. Certain restrictions are now required. The disturbance $u(t)$ is assumed to be a stationary random signal and the correlation function $\phi_{uu}(\tau)$ is assumed to be equal to the sequence correlation function $\phi^s{}_{uu}(\tau)$, that is,

$$\phi_{uu}(\tau) = \lim_{T_0 \to \infty} \frac{1}{2T_0} \int_{-T_0}^{T_0} u(t)u(t+\tau)dt$$

$$= \phi^s{}_{uu}(\tau) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} u(nT)u(nT+\tau) \qquad (2)$$

It is also assumed that $u(t)$ and $r(t)$ are statistically independent so that the cross correlation functions $\phi_{ru}(\tau)$ and $\phi_{ur}(\tau)$ are both zero. The Fourier transform of $\phi_{uu}(\tau)$ is called the spectral density of $u(t)$ and the controller design may be effected using spectral densities. Further discussions may be found in Kuo (1963) and Kloubec (1971).

The control problem can be stated as follows. For a process described by $G_u(s)$ and $G_s(s)$ with known input spectral densities $\Phi_{uu}(s)$ and $\Phi_{rr}(s)$, design the discrete controllers $D_u(z)$ and $D_r(z)$ such that the mean-square error $\overline{e^2(t)}$ is minimized subject to the mean square constraints $m_i^2(t) \leq W_i$, $i = 1, 2, \ldots I$. $M_i(t)$ is defined by

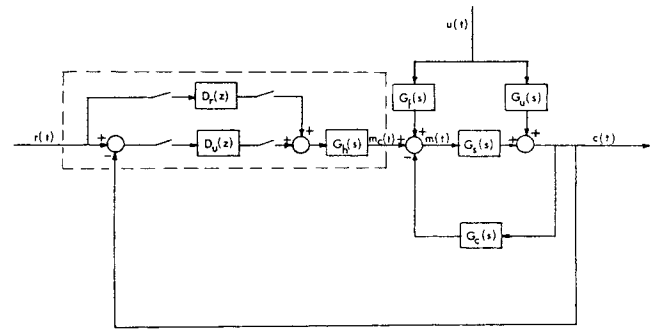

Fig. 1. Closed-loop DDC system.

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$M_i(s) = B_i(s)M(s)$ so that many types of constraints can be handled by proper selection of $B_i(s)$.

Using the method of Lagrange multipliers, the system cost function J may be defined as

$$J = \overline{e^2(t)} + \sum_{i=1}^{I} \lambda_i \overline{m_i^2(t)} \tag{3}$$

which is to be minimized by picking the optimal $D_u(z)$ and/or $D_r(z)$ and then again minimized in terms of λ_i subject to $m_i^2(t) \leq W_i^2$ (Chang, 1961).

OPTIMIZATION

It is first necessary to express the cost function J in terms of the system transfer functions and the spectral densities of the inputs. In terms of spectral densities Equation (3) is

$$J[D_u(z), D_r(z); \lambda_i] = \frac{1}{2\pi j} \oint_\Gamma \left[\Phi_{ee}(z) + \sum_{i=1}^{I} \lambda_i \Phi_{ii}(z) \right] \frac{dz}{z} \tag{4}$$

where $\Phi_{ee}(z)$ and $\Phi_{ii}(z)$ are the pulse-spectral densities of $\Phi_{ee}(s)$ and $\Phi_{m_im_i}(s)$, respectively. The contour integration is performed on Γ, the unit circle in the z-plane, in a counterclockwise direction. Now, $\Phi_{ee}(z)$ and $\Phi_{ii}(z)$ are expressed in terms of the spectral densities of the inputs $\Phi_{rr}(z)$ and $\Phi_{uu}(z)$, the system transfer functions G_c, G_r, G_h, G_s, and G_u, and the digital controllers D_r and D_u. Using the method proposed by Chang (1961), the result is derived and given by Kloubec (1971). Then, the problem reduces to that of determining the optimal $D_r(z)$ and $D_u(z)$ which minimize J. Certain system restrictions must be imposed prior to the determination of the optimal $D_r(z)$ and $D_u(z)$. They are:

1. If G_c is present, it must be so selected that not only G_c but also $G_s(s)/(1 + G_s(s)G_c(s))$ is stable.
2. Any pole of $G_u(s)$ outside of the left half s-plane (LHP) must be present in $G_s(s)$.

The stability of $E(s)$ and $M_i(s)$ will thus be assured.

The optimal design quantities $\hat{D}_r(z)$ and $\hat{D}_u(z)$ may be determined by minimizing J through the Wiener-Hopf technique (Chang, 1961). The procedure involved is equivalent to obtaining the Euler-Lagrange equations in the frequency domain. As derived by Koubec (1971) the necessary and sufficient conditions equivalent to two Euler-Lagrange equations in the frequency domain lead to

$$\hat{D}_u(z) = \hat{W}_u(z)/[1 - \hat{W}_u(z)G_0(z)] \tag{5}$$

and

$$\hat{D}_r(z) = [\hat{W}_r(z)/\hat{W}_u(z) - 1]\,\hat{D}_u(z) \tag{6}$$

where

$$W_i(z) = N_i(z)/[\Psi_i(z)K(z)] \qquad i = r, u \tag{7}$$

$$\Psi_i(z)\,\Psi_i(z^{-1}) = \Psi_{ii}(z) \qquad i = r, u \tag{8}$$

$$K(z)\,K(z^{-1}) = K_{ru}(z) \tag{9}$$

$$N_i(z) = z\{L_i(z)/[z\Psi_i(z^{-1})K(z^{-1})]\}_+ \qquad i = r, u \tag{10}$$

$$\Psi_{rr}(z) = \Phi_{rr}(z) \quad L_r(z) = \overline{G_0}\Psi_{rr}(z) \tag{11}$$

$$\Psi_{uu}(z) = \overline{G_d}G_d\Phi_{uu}(z) + \overline{G_d}G_fG_p\Phi_{uu}(z) + G_d\overline{G_f}\overline{G_p}\Phi_{uu}(z) + \overline{G_f}\overline{G_p}G_fG_p\Phi_{uu}(z) \tag{12}$$

$$L_u(z) = \overline{G_0}\Psi_{uu}(z) + \sum_{i=1}^{I} \lambda_i[\overline{B_i}B_i\overline{G_0}\overline{G_c}G_c\Psi_{uu}(z) - \overline{B_i}B_i\overline{G_h}G_c\Psi_{uu}(z) - \overline{B_i}B_iG_f\overline{G_d}\overline{G_0}\overline{G_c}\Phi_{uu}(z) - \overline{B_i}B_i\overline{G_f}G_f\overline{G_p}\overline{G_0}\overline{G_c}\Phi_{uu}(z) + \overline{B_i}B_iG_f\overline{G_d}\overline{G_h}\Phi_{uu}(z) + \overline{B_i}B_i\overline{G_f}G_f\overline{G_0}\Phi_{uu}(z)] \tag{13}$$

$$K_{ru}(z) = \overline{G_0}G_0(z) + \sum_{i=1}^{I} \lambda_i[\overline{B_i}B_i\overline{G_0}G_0\overline{G_c}G_c(z) - \overline{B_i}B_i\overline{G_0}\overline{G_c}G_h(z) - \overline{B_i}B_iG_0G_c\overline{G_h}(z) + \overline{B_i}B_i\overline{G_h}G_h(z)] \tag{14}$$

$$G_p(s) = G_s(s)/[1 + G_s(s)G_c(s)], \qquad G_d(s) = G_u(s)/[1 + G_s(s)G_c(s)] \tag{15}$$

and

$$G_0(s) = G_h(s)G_p(s) \tag{16}$$

Equation (8) represents spectral factorization so that $\Psi_i(z)$ has poles and zeroes inside the unit circle and $\Psi_i(z^{-1})$ has poles and zeroes outside the unit circle. In Equation (10) the notation $\{\ \}_+$ denotes the stable partial fraction of the quantity in the brackets, that is,

$$\frac{L(z)}{z\Psi(z^{-1})K(z^{-1})} = \left\{\frac{L(z)}{z\Psi(z^{-1})K(z^{-1})}\right\}_+ + \left\{\frac{L(z)}{z\Psi(z^{-1})K(z^{-1})}\right\}_- + V(z) \tag{17}$$

where $\left\{\frac{L(z)}{z\Psi(z^{-1})K(z^{-1})}\right\}_+$ contains all poles inside or on the unit circle except the poles of $1/[z\Psi(z^{-1})K(z^{-1})]$ on the unit circle, $\left\{\frac{L(z)}{z\Psi(z^{-1})K(z^{-1})}\right\}_-$ contains all finite poles outside the unit circle and the poles of $1/[\Psi(z^{-1})K(z^{-1})]$ on the unit circle, and $V(z)$ is a polynomial in z.

APPLICATION TO A CONTINUOUS STIRRED-TANK CHEMICAL REACTOR

It is desired to design an optimal digital regulator which will manipulate the cooling water flow rate Q_c of a continuous stirred-tank chemical reactor (CSTR) in order to compensate for the effect of fluctuations in the inlet concentration x_i on the reactor concentration x. The linearized system representation as given by Kloubec (1971) is

$$\bar{x}(s) = \frac{5.000 \times 10^{-3}(s - 1.549}{(s + 3.555 \times} \times 10^{-2})\,\bar{x}_i(s) + 6.930 \times 10^{-6}\,\bar{Q}_c(s) \over 10^{-3})(s - 5.919 \times 10^{-3}) \tag{18}$$

Note that the reactor is unstable and nonminimum phase. In order to stabilize the reactor the temperature was fed back via a proportional controller with gain $K_a = 0.3009$. This was equivalent, in terms of controller design, to specifying auxiliary controllers $G_c(s) = 58.70(1 + 76.18s)$ and $G_f(s) = 22.36$.

Two different types of disturbance in x_i were used. The spectral density of the disturbance was given by

$$\Phi_{\bar{x}_i\bar{x}_i}(s) = 4 \times 10^{-7}/[(-s + 2 \times 10^{-3})(s + 2 \times 10^{-3})] \tag{19}$$

which represents a deterministic series of alternating steps or random noise resulting from putting white noise through a first-order delay element. The constraint in this case was a magnitude constraint on the cooling water flow rate, $|Q_c| \leq 0.6$. This quantity was converted to a mean-square constraint using the technique developed by Kloubec (1971), $\overline{m_1^2(t)} = \overline{m^2(t)} \leq 0.04$.

Both the nonlinear stirred-tank reactor model and its linear approximation were controlled in the presence of six different disturbances, which consisted of each of the previously discussed types of disturbance taken at 0.5, 1.0, and 1.5 times the design amplitude. Five different optimal controllers were used, one at each of the sampling periods 5, 10, 20, 30, and 40 sec. A sampling rate of once every 10 sec. seemed to be optimal in this case. The digital control algorithm from Equation (5) was

$$M_1 m_c(nT) = C_1 c(nT) + C_2 c([n-1]T) + C_3 c([n-2]T) - M_2 m_c([n-1]T) - M_3 m_c([n-2]T) \quad (20)$$

The optimal controller coefficients for $T = 10$ sec. were $M_1 = 1.000$, $M_2 = -4.242 \times 10^{-1}$, $M_3 = -5.447 \times 10^{-1}$, $C_1 = -2.317 \times 10^3$, $C_2 = 3.392 \times 10^3$ and $C_3 = -1.209 \times 10^3$.

For the nonlinear reactor as well as the linearized model there appeared to be no perceptible offset for any disturbance, and for alternating step disturbances, the settling times were short while the peak overshoots were not excessive. Fluctuations in the stochastic disturbances were essentially absent in the outputs. The linearized reactor responded to step disturbances with less error than the nonlinear reactor, but the responses of the two systems to stochastic disturbances were essentially equivalent.

For comparison purposes, the optimal continuous controller designed by Lim and Bankoff (1970) was used to control the system. The response of this continuous compensation is compared in Figure 2 with the response with the optimal digital controller with a 10 sec. sampling period. There appear to be no significant differences.

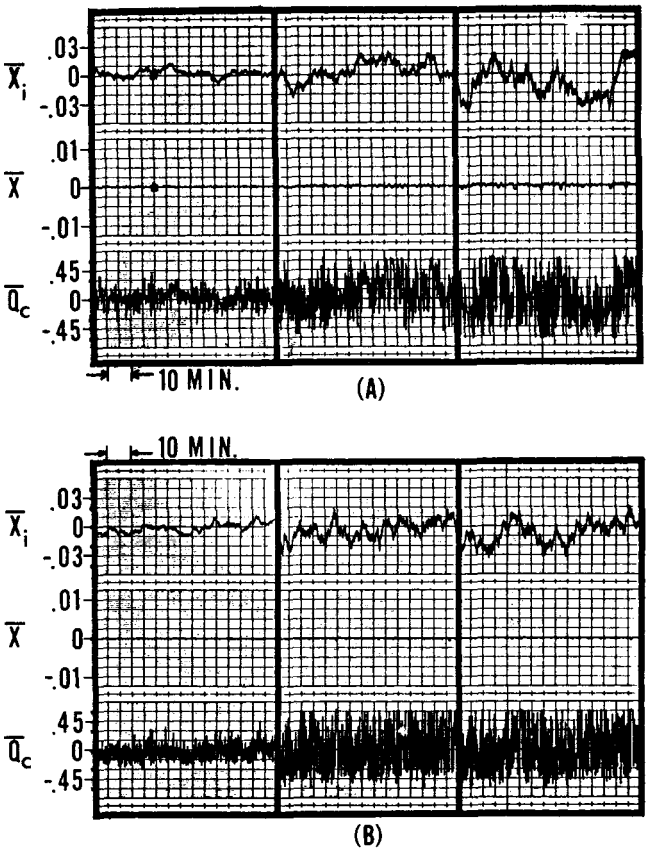

Fig. 2. Control of chemical reactor subject to random disturbances. (A) DDC. (B) continuous control.

CONCLUSIONS AND SUMMARY

Using Wiener-Hopf techniques, mathematical expressions for optimal digital control of a continuous linear system were developed which are valid for a very general linear, time-invariant lumped-parameter system allowing for unstable, nonminimum phase and dead time behavior. The inputs to the system were stationary stochastic signals whose probability distributions were known.

The solution developed above was used to design the optimal digital regulator to control a CSTR about its unstable steady state. Both the nonlinear CSTR model and its linear approximation were simulated on the analog portion of a hybrid computer, and the control algorithm was implemented on the digital portion. The performance at $T = 10$ sec. was essentially equivalent to that of a system compensated by a continuous controller using Wiener-Hopf techniques similar to those used in this work.

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NOTATION

$D_r(z)$ = set point digital controller
$D_u(z)$ = disturbance digital controller
$G_c(s)$ = auxiliary continuous feedback controller
$G_h(s)$ = zero-order hold transfer function
$G_f(s)$ = auxiliary continuous feedforward controller
$G_s(s)$ = plant transfer function
$G_u(s)$ = disturbance transfer function
$c(t), c(s)$ = system output
$m(t), M(s)$ = manipulated variable
$m_c(t)$ = combined digital controller output
$r(t), R(s)$ = system set point
$u(t), U(s)$ = system disturbance
τ_s, τ_u = deadtime associated with $G_s(s)$ and $G_u(s)$ respectively

Superscripts

$\wedge$ = optimal quantity as $W_r(z)$
$\overline{\quad}$ = mean-square value as $\overline{x^2(t)}$
$-$ = complex conjugate as $\overline{G}(s) = G(-s)$
$-$ = deviation variable as $\overline{Q}_c(t)$

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Effect of Axial Dispersion on Microbial Growth

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The extent of microbial growth in a continuous flow reactor is influenced by various degrees of fluid mixing which may range from one extreme of plug-flow to the other extreme of completely mixed flow. To assess the effect of mixing on the extent of microbial growth various flow models such as the tanks-in-series model and the axial dispersion model are often employed. Such flow models have been successfully applied to biological waste treatment processes by Murphy (1971) and Lee et al. (1971).

The purpose of this note is to present and discuss the analytical solutions of differential equations describing the extent of microbial growth based on the axial dispersion model under the assumption that the specific growth rate does not depend on the substrate concentration. The assumption permits the differential equations to be solved analytically and provides useful limiting case for more complicated substrate limited systems. These results should be useful in connection with fermentation processes and biological waste water treatment.

Specifically two problems are considered. The first is concerned with steady state solution with nonsterile feed. The second is concerned with the transient and steady state solution with sterile feed.

STEADY STATE GROWTH WITH NONSTERILE FEED [$X(0) > 0$]

The differential equation based on the axial dispersion model for this case can be represented in the following dimensionless form with Danckwerts' boundary conditions:

$$\frac{1}{Pe}\frac{d^2X}{dz^2} - \frac{dX}{dz} + KX = 0 \tag{1}$$

$$X - \frac{1}{Pe}\frac{dX}{dz} = 1 \qquad \text{at} \quad z = 0 \tag{2}$$

$$\frac{dX}{dz} = 0 \qquad \text{at} \quad z = 1 \tag{3}$$

where Pe is the Peclet number. Pe is a measure of the degree of mixing, ranging from $Pe = 0$ (completed mixed flow) to $Pe = \infty$ (plug flow).

The solution can be obtained by a standard method such as Laplace transformation for the following three cases:

(i) $Pe > 4K$

$$X(z) = \frac{2\exp\left(\frac{Pe\,z}{2}\right)\left[\sinh\frac{aPe}{2}(1-z) + a\cosh\frac{aPe}{2}(1-z)\right]}{(1+a^2)\sinh\frac{aPe}{2} + 2a\cosh\frac{aPe}{2}} \tag{4}$$

where

$$a \equiv \sqrt{1 - \frac{4K}{Pe}}$$

(ii) $Pe = 4K$

$$X(z) = \frac{1 + \frac{Pe}{2}(1-z)}{1 + \frac{Pe}{4}}\exp\left(\frac{Pe\,z}{2}\right) \tag{5}$$